BSML and Expressive Completeness

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NihiL seminar
ILLC, University of Amsterdam

Outline for the talk

- Background and Motivation
- Preliminaries
    - What is **BSML**?
    - What is expressive completeness?
- Expressive Completeness of **BSML**
- Convex Team Logic
- Conclusion

Background and Motivation

First NihiL talk:

- Among more, Aleksi introduced two extensions of **BSML**, and proved that they were expressively completely for all properties [invariant under bounded bisimulation] and all union-closed properties, respectively.
- The problem of characterizing the expressive power of **BSML** was left open.

Today (last NihiL talk before Summer hiatus):

- We show that **BSML** is expressively complete for all convex, union-closed properties.
- We introduce a logic which is expressively complete for all convex properties simpliciter.

Why expressive completeness?

- Characterization of logic (à la van Benthem)
- Provides normal form
- Normal form as heuristic for proof theory

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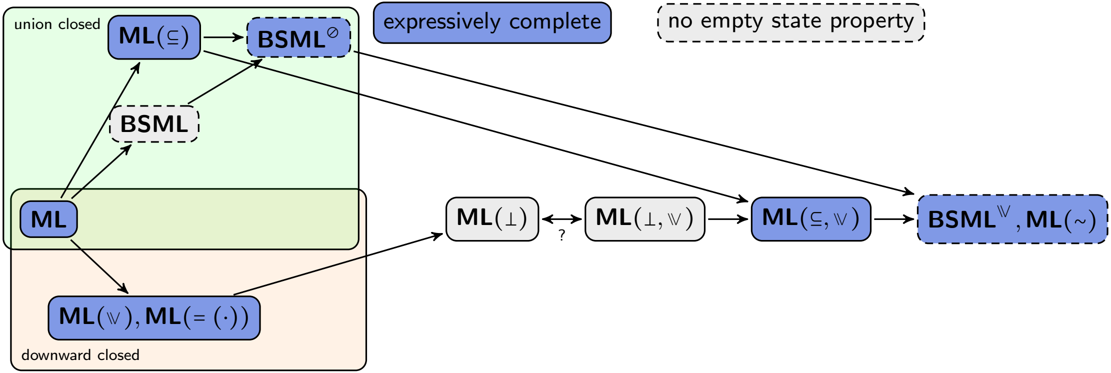
Expressive powers compared:



Syntax of **BSML**

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \Diamond \phi \mid \mathrm{NE}$$

Semantics for support ($\models$)

$$
\begin{aligned}
s \models p &\iff \forall w \in s : w \in V(p)\\
s \models \neg\phi &\iff s \dashv \phi\\
s \models \phi \wedge \psi &\iff s \models \phi \text{ and } s \models \psi\\
s \models \phi \vee \psi &\iff \exists t, t' : t \cup t' = s \text{ and } t \models \phi \text{ and } t' \models \psi\\
s \models \Diamond\phi &\iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \models \phi\\
s \models \mathrm{NE} &\iff s \neq \varnothing
\end{aligned}
$$

$R[w] = \{v \in W \mid wRv\}$

Syntax of **BSML**

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \Diamond \phi \mid \mathrm{NE}$$

Semantics for support ($\vDash$)

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s \vDash p &\iff \forall w \in s : w \in V(p)\\
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s \vDash \Diamond\phi &\iff \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \vDash \phi\\
s \vDash \mathrm{NE} &\iff s \neq \varnothing
\end{aligned}$$

$$\boxed{R[w] = \{v \in W \mid wRv\}}$$

Split disjunction '$\vee$' and the non-emptiness atom 'NE'

$$s \vDash \phi \vee \psi \iff \exists t, t' : \quad t \cup t' = s, t \vDash \phi,\ t' \vDash \psi$$
$$s \vDash \mathrm{NE} \iff s \neq \varnothing$$

(a) $s \vDash (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$ (b) $s \nvDash (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

Expressive Completeness

OBS: In what follows, we fix a *finite* set of propositional letters **P**.

Definition

- A *pointed state model* is a pair (M, s) where M is a model over **P** and s is a state on M.
- A *(state) property* is a class of pointed state models $\{(M, s)\}$.
- For a formula ϕ, we define its state property as $||\phi|| := \{(M, s) \mid M, s \vDash \phi\}$.

Definition (Expressive Completeness)

We say that a logic (or language) $\mathcal{L}$ is *expressively complete* for a class of properties $\mathcal{C}$:iff

$$||\mathcal{L}|| := \{||\phi|| \mid \phi \in \mathcal{L}\} = \mathcal{C}$$

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Definition (Closure properties)

We say that

ϕ is *downward closed*	**iff**	$[M, s \models \phi$ and $t \subseteq s] \implies M, t \models \phi$
ϕ is *union closed*	**iff**	$[M, s \models \phi$ for all $s \in S \neq \varnothing] \implies M, \bigcup S \models \phi$
ϕ has the *empty state property*	**iff**	$M, \varnothing \models \phi$ for all M
ϕ is *flat*	**iff**	$M, s \models \phi \iff M, \{w\} \models \phi$ for all $w \in s$

Observe: flat $\iff$ downward closed & union closed & empty state property

And observe: For formulas α in classical modal logic **ML** (no NE):

$$s \models \alpha \iff \forall w \in s : \{w\} \models \alpha \iff \forall w \in s : w \models \alpha$$

Proposition

$$\{||\phi|| \mid \phi \in \mathbf{ML}\} = \{\text{property } \mathcal{P} \mid \mathcal{P} \text{ is flat and invariant under bounded bisimulation }\}$$

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State k-bisimulation:

$s \rightleftharpoons_k s' :\iff$
forth: $\forall w \in s : \exists w' \in s' : w \rightleftharpoons_k w'$
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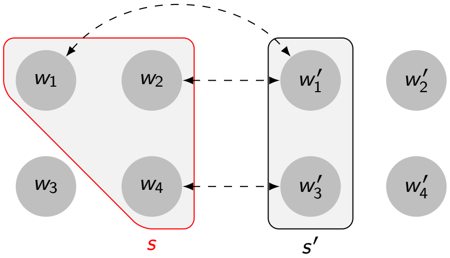
Observe: $s \rightleftharpoons_k s' \implies s \equiv^k s'$



Definition

We say that a property $\mathcal{P}$ is *invariant under bounded bisimulation* :iff it is invariant under k-bisimulation for some $k \in \omega$.

Fact

Restricting to our finite set of propositional letters **P**, for any world $w \in M$, we can define Hintikka formulas $\chi_w^k \in$ **ML** s.t. for all w':

$$w' \vDash \chi_w^k \iff w \rightleftharpoons_k w'$$

Thus, for any team t, we can define formulas $\chi_t^k := \bigvee_{w \in t} \chi_w^k$ s.t.

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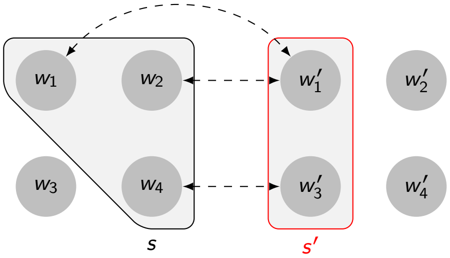

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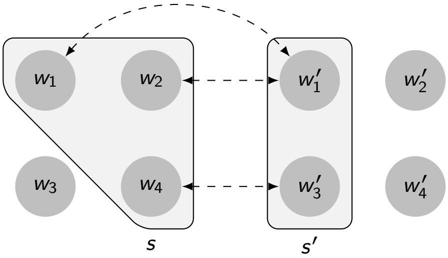

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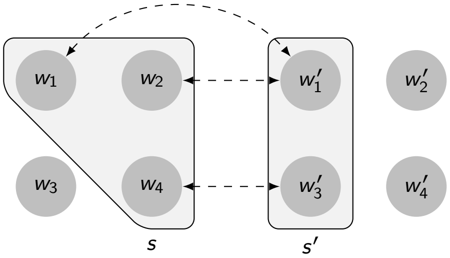

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Theorem (Aloni, Anttila, Yang [2023])

$$\|\mathcal{BSML}^{\mathbb{V}}\| = \{\text{property } \mathcal{P} \mid \mathcal{P} \text{ is invariant under bounded bisimulation}\}$$

and

$$\|\mathcal{BSML}^{\oslash}\| = \{\text{property } \mathcal{P} \mid \mathcal{P} \text{ is union closed and invariant under bounded bisimulation}\}$$

Definition

We say that a formula ϕ is *convex* :iff

$$\text{if } t \models \phi, t'' \models \phi \text{ and } t \subseteq t' \subseteq t'', \text{ then } t' \models \phi.$$

Theorem (expressive completeness of BSML)

$$\|\mathcal{BSML}\| = \{\text{property } \mathcal{P} \mid \mathcal{P} \text{ is convex, union closed and invariant under bounded bisimulation}\}$$

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Proof.

'⊆': Bounded bisimulation: ✓

Union closure: ✓

Convexity: By induction, *see blackboard*

'⊇': Let $\mathcal{P}$ be an arbitrary convex, union closed property invariant under k-bisimulation.

- If there is some $(M, \varnothing) \in \mathcal{P}$, then by invariance under k-bisimulation, $\mathcal{P}$ has the empty state property. So by convexity, it is downwards closed, hence flat. Thus, we can find $\phi \in \mathbf{ML} \subseteq \mathbf{BSML}$ s.t. $||\phi|| = \mathcal{P}$.
- If not, take representatives $t_1, \ldots, t_n$ of k-bis. equivalence classes and consider the following formula:

$$\varphi^k_{\mathcal{P}} := \bigvee \left(\left\{ NE \wedge (\chi^k_{w_1} \vee \cdots \vee \chi^k_{w_n}) \mid (w_1, \ldots, w_n) \in (t_1 \times \cdots \times t_n) \right\} \right)$$

We claim that $||\varphi^k_{\mathcal{P}}|| = \mathcal{P}$. *See blackboard* □

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Theorem (expressive completeness of BSML)

$$||\mathcal{BSML}|| = \{\text{property } \mathcal{P} \mid \mathcal{P} \text{ is convex, union closed and invariant under bounded bisimulation}\}$$

Proof.

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Union closure: ✓

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Recap and normal form

- We have shown that **BSML** is expressively complete for all convex, union-closed properties.
- We have obtained a normal form for **BSML**-formulas ϕ, namely of the form

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- Or, in fact, equivalently:

$$\bigvee_{t \in \mathcal{P}} \chi^k_t \wedge \bigwedge \{ ((\chi^k_{w_1} \vee \chi^k_{w_2} \vee \ldots \vee \chi^k_{w_n}) \wedge \mathrm{NE}) \vee \top \mid (w_1, \ldots, w_n) \in (t_1 \times \cdots \times t_n) \},$$

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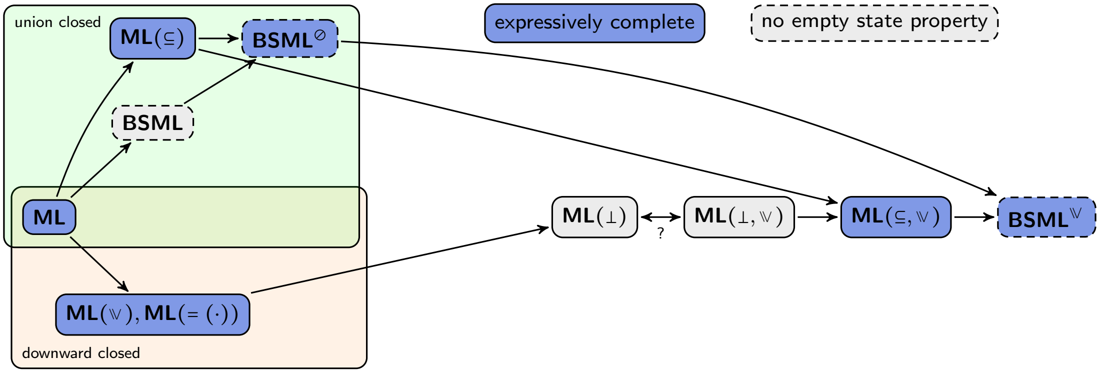
Updated picture:



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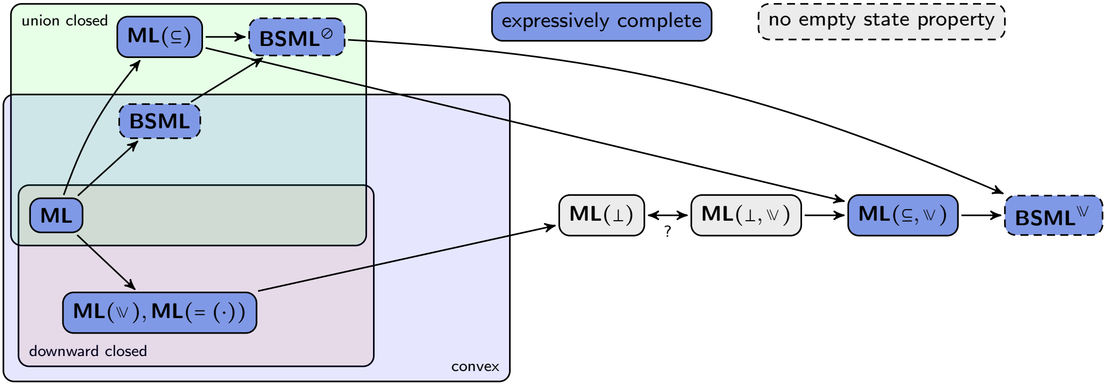

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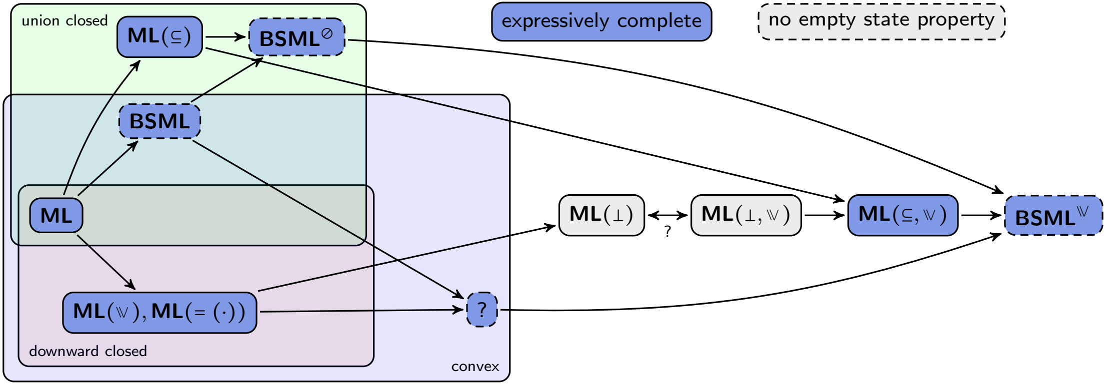

What logic is expressively complete for convex properties (without the empty team property)? Note:

$$\phi \text{ is convex and has the empty team property} \iff$$
$$\phi \text{ is downward closed and has the empty team property}$$

So $\mathbf{ML}(=(\cdot))$ is expressively complete for convex properties with the empty team property.

Examples of convex sentences/formulas which are not union closed:

Between five and ten bananas are yellow.

$(q \mathbin{\mathbb{V}} \neg q) \wedge ((r \wedge \mathrm{NE}) \vee \mathbb{T})$ (where $\mathbb{T} := (p \vee \neg p)$)

Recall the following characteristic formulas for convex union-closed properties:

If $\mathcal{P} \neq \varnothing$: $$\bigvee_{s \in \mathcal{P}} \chi^k_s \wedge \bigwedge \{((\chi^k_{w_1} \vee \chi^k_{w_2} \vee \ldots \vee \chi^k_n) \wedge \mathrm{NE}) \vee \top\!\!\top \mid (w_1, \ldots, w_n) \in (s_1 \times \cdots \times s_n)\}$$

which we may write: $$\bigvee_{s \in \mathcal{P}} \chi^k_s \wedge \bigwedge_{u \in \prod \mathcal{P}} ((\chi^k_u \wedge \mathrm{NE}) \vee \top\!\!\top)$$

If $\mathcal{P} = \varnothing$: $$\bot \wedge \mathrm{NE}$$

where $\mathcal{P} \rightleftharpoons_k \{s_1, \ldots, s_n\}$. The first conjunct in the non-empty characteristic formula is a characteristic formula for flat properties, and the second for upward-closed properties.

Recall the following characteristic formulas for convex union-closed properties:

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To get a characteristic formula for (non-empty) convex properties, simply replace the first conjunct with a characteristic formula for downward-closed properties:

$$\mathop{\bigvee\!\!\!\!\bigvee}_{s \in \mathcal{P}} \chi_s^k \wedge \bigwedge_{u \in \prod \mathcal{P}} ((\chi_u^k \wedge \mathrm{NE}) \vee \top\!\!\!\top)$$

Proposition

For any non-empty convex $\mathcal{P}$ invariant under $\rightleftharpoons_k$:

$$t \in \mathcal{P} \iff t \vDash \bigvee_{s \in \mathcal{P}} \chi_s^k \wedge \bigwedge_{u \in \prod \mathcal{P}} ((\chi_u^k \wedge \mathrm{NE}) \vee \top\!\!\top)$$

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Proof.

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For any non-empty convex $\mathcal{P}$ invariant under $\rightleftharpoons_k$:

$$t \in \mathcal{P} \iff t \models \bigvee_{s \in \mathcal{P}} \chi_s^k \wedge \bigwedge_{u \in \prod \mathcal{P}} ((\chi_u^k \wedge \mathrm{NE}) \vee \top)$$

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Claim: by $t \vDash \bigwedge_{u \in \prod \mathcal{P}} ((\chi_u^k \wedge \mathrm{NE}) \vee \top\!\!\top)$ there is some $y \in \mathcal{P}$ s.t. $y \rightleftharpoons_k t' \subseteq t$. Assume for contradiction that $\forall s \in \mathcal{P} : \exists w_s \in s : \nexists v \in t : w_s \rightleftharpoons_k v$ (i.e., $\forall s \in \mathcal{P} : s \notin t$, in modal terms).

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For any non-empty convex $\mathcal{P}$ invariant under $\rightleftharpoons_k$:

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Claim: by $t \vDash \bigwedge_{u \in \prod \mathcal{P}} ((\chi_u^k \wedge \mathrm{NE}) \vee \top\!\!\!\top)$ there is some $y \in \mathcal{P}$ s.t. $y \rightleftharpoons_k t' \subseteq t$. Assume for contradiction that $\forall s \in \mathcal{P} : \exists w_s \in s : \nexists v \in t : w_s \rightleftharpoons_k v$ (i.e., $\forall s \in \mathcal{P} : s \notin t$, in modal terms). Then $\{w_s \mid s \in \mathcal{P}\} \in \prod \mathcal{P}$ so $t \vDash ((\bigvee_{\{w_s \mid s \in \mathcal{P}\}} \chi_{w_s}^k) \wedge \mathrm{NE}) \vee \top\!\!\!\top$. But then for some $s \in \mathcal{P}$ we have $t \vDash (\chi_{w_s}^k \wedge \mathrm{NE}) \vee \top\!\!\!\top$ so for some $v \in t : w \rightleftharpoons_k v$, a contradiction. So for some $y \in \mathcal{P}$ we must have $\forall w \in y : \exists v \in t : w \rightleftharpoons_k v$, i.e., $y \rightleftharpoons_k t' \subseteq t$.

Proposition

For any non-empty convex $\mathcal{P}$ invariant under $\rightleftharpoons_k$:

$$t \in \mathcal{P} \iff t \models \bigvee_{s \in \mathcal{P}} \chi_s^k \wedge \bigwedge_{u \in \prod \mathcal{P}} ((\chi_u^k \wedge \mathrm{NE}) \vee \top\!\!\top)$$

Proof.

$\Longrightarrow$: Clearly $t \models \chi_t^k$ so $t \models \bigvee_{s \in \mathcal{P}} \chi_s^k$. If $\mathcal{P}$ has the empty team property, $\prod \mathcal{P} = \varnothing$ so the second conjunct is $\top\!\!\top$ and we are done. Otherwise let $u \in \prod \mathcal{P}$. For some $w \in u$ we have $w \in t$, so $t \models (\chi_w^k \wedge \mathrm{NE}) \vee \top\!\!\top$. Therefore also $t \models ((\chi_w \vee \bigvee_{v \in u \setminus \{w\}} \chi_v^k) \wedge \mathrm{NE}) \vee \top\!\!\top$ whence $t \models (\chi_u^k \wedge \mathrm{NE}) \vee \top\!\!\top$.

$\Longleftarrow$: By $t \models \bigvee_{s \in \mathcal{P}} \chi_s^k$ there is some $s \in \mathcal{P}$ s.t. $t \rightleftharpoons_k s' \subseteq s$.
Claim: by $t \models \bigwedge_{u \in \prod \mathcal{P}} ((\chi_u^k \wedge \mathrm{NE}) \vee \top\!\!\top)$ there is some $y \in \mathcal{P}$ s.t. $y \rightleftharpoons_k t' \subseteq t$. Assume for contradiction that $\forall s \in \mathcal{P} : \exists w_s \in s : \nexists v \in t : w_s \rightleftharpoons_k v$ (i.e., $\forall s \in \mathcal{P} : s \notin t$, in modal terms). Then $\{w_s \mid s \in \mathcal{P}\} \in \prod \mathcal{P}$ so $t \models ((\bigvee_{\{w_s \mid s \in \mathcal{P}\}} \chi_{w_s}^k) \wedge \mathrm{NE}) \vee \top\!\!\top$. But then for some $s \in \mathcal{P}$ we have $t \models (\chi_{w_s}^k \wedge \mathrm{NE}) \vee \top\!\!\top$ so for some $v \in t : w \rightleftharpoons_k v$, a contradiction. So for some $y \in \mathcal{P}$ we must have $\forall w \in y : \exists v \in t : w \rightleftharpoons_k v$, i.e., $y \rightleftharpoons_k t' \subseteq t$.
By $\rightleftharpoons_k$-invariance, $t' \in \mathcal{P}$. $t' \subseteq t \rightleftharpoons_k s'$, so $t' \rightleftharpoons_k s'' \subseteq s'$ whence $s'' \in \mathcal{P}$ by $\rightleftharpoons_k$-invariance. $s'' \subseteq s' \subseteq s \in \mathcal{P}$ so $s' \in \mathcal{P}$ by convexity. Then $t \in \mathcal{P}$ by $\rightleftharpoons_k$-invariance. □

So we can capture all convex properties in $\mathbf{ML}(\mathrm{NE}, \mathbb{V})$, but this is clearly not convex; e.g., $((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})) \mathbb{V}\, q$ is not convex.

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This is not surprising given $\mathbf{ML}(\mathrm{NE}, \mathbb{V})$ is complete for all properties, but there is a more general issue with the tensor disjunction: if ϕ or ψ is not union closed, $\phi \vee \psi$ might not be convex:

Fact

If a logic can express all convex properties and has the connective $\vee$, it is not convex.

Recall the intuitionistic implication $\rightarrow$:

$$s \vDash \phi \rightarrow \psi \iff \forall t \subseteq s : t \vDash \phi \text{ implies } t \vDash \psi$$

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Consider $\psi := (((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})) \rightarrow q) \wedge ((r \wedge \mathrm{NE}) \vee \top)$. It is easy to see that $\|\psi\|$ is convex (the first conjunct is downward closed; the second, upward closed) and not union closed.

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Recall the intuitionistic implication $\to$:

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To obtain an expressively complete convex logic, we change the classical base of the logic.

Syntax of classical modal logic with → **ML**$_{\to}$:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \alpha \to \alpha \mid \Diamond \alpha$$

Syntax of modal convex team logic **MC**:

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∇ is the "epistemic might" operator which has been used to formalize epistemic contradictions:

$$s \models \nabla \phi \iff \exists t \subseteq s : t \neq \varnothing \text{ and } t \models \phi$$

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Epistemic contradiction: #It is raining but it might not be raining.
Formalized as: $r \wedge \nabla \neg r$. Contradiction: $r \wedge \nabla \neg r \models \bot\!\!\!\bot$.

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Note that $\nabla\phi \equiv (\phi \wedge \mathrm{NE}) \vee \top\!\!\!\top$ and that $\mathrm{NE} \equiv \nabla \top\!\!\!\top$.

Proposition

MC is convex.

Proof.

$p, \bot$ and $\Diamond\phi$ are flat and and hence convex. $\phi \to \phi$ is downward closed and hence convex. $\nabla\phi$ is upward closed and hence convex. The conjunction case follows immediately from the induction hypothesis. □

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By the foregoing, if **MC** can express the empty property, all upward-closed properties, and all downward-closed properties, it can express all convex properties.

MC can express the empty property since $t \in \mathcal{P} \iff t \vDash \nabla\bot$.

MC can express all upward-closed properties since

$$\bigwedge_{u \in \prod \mathcal{P}} ((\chi_u^k \wedge \mathrm{NE}) \vee \top) \equiv \bigwedge_{u \in \prod \mathcal{P}} \nabla\chi_u^k$$

To show **MC** can express all downward-closed properties, we show that the global disjunction is definable for classical formulas. For $\{\alpha\}_{i \in I} \subseteq \mathbf{ML}_{\rightarrow}$ define:

$$\bigvee\!\!\!\bigvee_{i \in I} \alpha_i := \bigwedge_{i \in I}((\bigwedge_{j \in I \setminus \{i\}} \nabla \neg \alpha_j) \rightarrow \alpha_i) \qquad \text{E.g., } \alpha \mathbin{\vee\!\!\!\vee} \beta = (\nabla \neg \alpha \rightarrow \alpha) \wedge (\nabla \neg \beta \rightarrow \beta)$$

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Lemma

$t \models \bigvee\!\!\!\bigvee_{i\in I}\alpha_i \iff \exists i \in I : t \models \alpha_i$.

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Lemma

$t \vDash \bigvee\!\!\!\!\bigvee_{i\in I}\alpha_i \iff \exists i \in I : t \vDash \alpha_i$.

Proof.

$\Longrightarrow$: Assume for contradiction that for all $i \in I$ there is some $v_i \in t$ with $v_i \vDash \neg\alpha_i$.

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Proof.

$\Longrightarrow$: Assume for contradiction that for all $i \in I$ there is some $v_i \in t$ with $v_i \models \neg\alpha_i$. Then for each $i \in I$: $t \models \nabla\neg\alpha_i$.

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Proof.

$\Longrightarrow$: Assume for contradiction that for all $i \in I$ there is some $v_i \in t$ with $v_i \vDash \neg \alpha_i$. Then for each $i \in I$: $t \vDash \nabla \neg \alpha_i$. By $t \vDash (\bigwedge_{j \in I \setminus \{i\}} \nabla \neg \alpha_i) \rightarrow \alpha_i$, we have $t \vDash \alpha_i$ for all $i \in I$, a contradiction.

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Proof.

$\implies$: Assume for contradiction that for all $i \in I$ there is some $v_i \in t$ with $v_i \vDash \neg\alpha_i$. Then for each $i \in I$: $t \vDash \nabla\neg\alpha_i$. By $t \vDash (\bigwedge_{j \in I \setminus \{i\}} \nabla\neg\alpha_i) \rightarrow \alpha_i$, we have $t \vDash \alpha_i$ for all $i \in I$, a contradiction. So for some $i \in I$ we must have have $t \vDash \alpha_i$.

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$\Longrightarrow$: Assume for contradiction that for all $i \in I$ there is some $v_i \in t$ with $v_i \vDash \neg\alpha_i$. Then for each $i \in I$: $t \vDash \nabla\neg\alpha_i$. By $t \vDash (\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_i) \rightarrow \alpha_i$, we have $t \vDash \alpha_i$ for all $i \in I$, a contradiction. So for some $i \in I$ we must have have $t \vDash \alpha_i$.

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To show **MC** can express all downward-closed properties, we show that the global disjunction is definable for classical formulas. For $\{\alpha\}_{i\in I} \subseteq \mathbf{ML}_{\rightarrow}$ define:

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$\Longrightarrow$: Assume for contradiction that for all $i \in I$ there is some $v_i \in t$ with $v_i \vDash \neg\alpha_i$. Then for each $i \in I$: $t \vDash \nabla\neg\alpha_i$. By $t \vDash (\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_i) \to \alpha_i$, we have $t \vDash \alpha_i$ for all $i \in I$, a contradiction. So for some $i \in I$ we must have have $t \vDash \alpha_i$.

$\Longleftarrow$: Let $t \vDash \alpha_i$. Let $s \subseteq t$ be such that $s \vDash \bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j$.

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$\Longrightarrow$: Assume for contradiction that for all $i \in I$ there is some $v_i \in t$ with $v_i \models \neg \alpha_i$.Then for each $i \in I$: $t \models \nabla \neg \alpha_i$. By $t \models (\bigwedge_{j \in I \setminus \{i\}} \nabla \neg \alpha_i) \rightarrow \alpha_i$, we have $t \models \alpha_i$ for all $i \in I$, a contradiction. So for some $i \in I$ we must have have $t \models \alpha_i$.

$\Longleftarrow$: Let $t \models \alpha_i$. Let $s \subseteq t$ be such that $s \models \bigwedge_{j \in I \setminus \{i\}} \nabla \neg \alpha_j$. By downward closure also $s \models \alpha_i$. So $t \models (\bigwedge_{j \in I \setminus \{i\}} \nabla \neg \alpha_j) \rightarrow \alpha_i$. Now fix $k \neq i; k \in I$. There can be no $s \subseteq t$ such that $s \models \bigwedge_{j \in I \setminus \{k\}} \nabla \neg \alpha_j$ because $s \models \alpha_i$.

To show **MC** can express all downward-closed properties, we show that the global disjunction is definable for classical formulas. For $\{\alpha\}_{i\in I} \subseteq \mathbf{ML}_{\to}$ define:

$$\mathop{\bigvee\!\!\!\!\bigvee}_{i\in I} \alpha_i := \bigwedge_{i\in I}\Big(\Big(\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j\Big) \to \alpha_i\Big) \qquad \text{E.g., } \alpha \mathbin{\bigvee\!\!\!\!\bigvee} \beta = (\nabla\neg\alpha \to \alpha) \wedge (\nabla\neg\beta \to \beta)$$

Lemma

$t \vDash \mathop{\bigvee\!\!\!\!\bigvee}_{i\in I}\alpha_i \iff \exists i \in I : t \vDash \alpha_i.$

Proof.

$\Longrightarrow$: Assume for contradiction that for all $i \in I$ there is some $v_i \in t$ with $v_i \vDash \neg\alpha_i$. Then for each $i \in I$: $t \vDash \nabla\neg\alpha_i$. By $t \vDash (\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_i) \to \alpha_i$, we have $t \vDash \alpha_i$ for all $i \in I$, a contradiction. So for some $i \in I$ we must have have $t \vDash \alpha_i$.

$\Longleftarrow$: Let $t \vDash \alpha_i$. Let $s \subseteq t$ be such that $s \vDash \bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j$. By downward closure also $s \vDash \alpha_i$. So $t \vDash (\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j) \to \alpha_i$. Now fix $k \neq i; k \in I$. There can be no $s \subseteq t$ such that $s \vDash \bigwedge_{j\in I\setminus\{k\}} \nabla\neg\alpha_j$ because $s \vDash \alpha_i$. Therefore $t \vDash (\bigwedge_{j\in I\setminus\{k\}} \nabla\neg\alpha_j) \to \alpha_k$. □

To show **MC** can express all downward-closed properties, we show that the global disjunction is definable for classical formulas. For $\{\alpha\}_{i\in I} \subseteq \mathbf{ML}_{\rightarrow}$ define:

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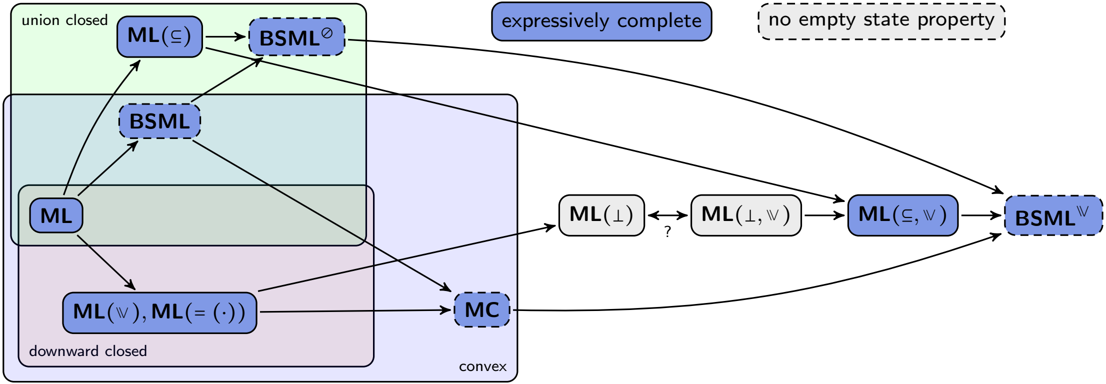
$\Longleftarrow$: Let $t \vDash \alpha_i$. Let $s \subseteq t$ be such that $s \vDash \bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j$. By downward closure also $s \vDash \alpha_i$. So $t \vDash (\bigwedge_{j\in I\setminus\{i\}} \nabla\neg\alpha_j) \rightarrow \alpha_i$. Now fix $k \neq i; k \in I$. There can be no $s \subseteq t$ such that $s \vDash \bigwedge_{j\in I\setminus\{k\}} \nabla\neg\alpha_j$ because $s \vDash \alpha_i$. Therefore $t \vDash (\bigwedge_{j\in I\setminus\{k\}} \nabla\neg\alpha_j) \rightarrow \alpha_k$. □

Theorem

MC *is complete for convex properties invariant under bounded bisimulation.*

Updated picture:



Relationship with inquisitive logic: Let **PC** be the propositional fragment of **MC**—syntax:

$$\phi ::= p \mid \bot \mid \phi \wedge \phi \mid \phi \to \phi \mid \nabla \phi$$

InqB, propositional inquisitive logic, has the syntax:

$$\phi ::= p \mid \bot \mid \phi \wedge \phi \mid \phi \to \phi \mid \phi \vee\!\!\!\vee \phi$$

InqB is expressively complete for downward-closed properties with the empty state property, so $\|InqB\| \subset \|\mathbf{PC}\|$. $\vee\!\!\!\vee$ is not definable in general in **PC** (since $\mathbf{PC} + \vee\!\!\!\vee$ is not convex).

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InqB is expressively complete for downward-closed properties with the empty state property, so $\|InqB\| \subset \|\mathbf{PC}\|$. $\mathbb{V}$ is not definable in general in **PC** (since $\mathbf{PC} + \mathbb{V}$ is not convex).

Similar logics which are either not convex or cannot express all convex properties (we consider propositional logics for simplicity):

> $PL_{\to}(\mathbb{V}, \nabla)$ (propositional inquisitive logic with ∇) is not convex. Example: $(p \wedge \nabla q) \mathbin{\mathbb{V}} (a \wedge \nabla b)$.

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Similar logics which are either not convex or cannot express all convex properties (we consider propositional logics for simplicity):

- $PL_{\rightarrow}(\mathbb{V}, \nabla)$ (propositional inquisitive logic with ∇) is not convex. Example: $(p \wedge \nabla q) \mathbin{\mathbb{V}} (a \wedge \nabla b)$.
- $PL_{\rightarrow}(\mathrm{NE})$ is not complete for convex properties because it is "downward closed except for the empty state": $s \models \phi$ and $t \subseteq s$ where $t \neq \varnothing$ imply $t \models \phi$. Similarly for $PL_{\rightarrow}(\mathrm{NE}, \mathbb{V})$.

Topics for further investigation:

Over formulas, dependence logic characterizes all downward closed Σ_1^1-properties. What logic characterizes all convex Σ_1^1-properties?

Are there any linguistic applications of convex team logic?

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